

# THE BILLIARD BALL PROBLEM AND ROTATION NUMBERS

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**ABSTRACT.** We introduce the concepts of rotation numbers and rotation vectors for billiard maps. Our approach is based on the birkhoff ergodic theorem. We anticipate that it will be useful, in particular, for the purpose of establishing the non-ergodicity of billiard in certain domains.

## 1. MOTIVATION

In [19] Poincare introduced and studied the concept of rotation number for orientation preserving circle homeomorphisms. Since then, rotation numbers and their generalizations became instrumental in several areas of mathematics. The purpose of a brief outline that follows is to give the reader an idea about the spread of the broadly interpreted concept of rotation number in the contemporary mathematical literature.

E. Ghys [5] used rotation numbers to show that certain classes of groups cannot act on the circle. A.I. Stern [21] encountered rotation numbers in the framework of quasi-representations of groups. M. Misiurewicz and others extended the framework of rotation numbers in several directions. In particular, rotation numbers become rotation sets when circle homeomorphisms are replaced by arbitrary continuous self-mappings of the circle or by homeomorphisms of multi-dimensional tori. We recommend the text [18] for a comprehensive exposition and an extensive bibliography.

The concept of rotation number is quite flexible which explains the differences between approaches to this concept in various dynamics frameworks [20]. Here we are interested in the uses of rotation numbers for billiard dynamics. But even in the subject of billiards there are several versions of the notion of rotation numbers. The work [3]

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defines rotation numbers in the framework of billiard flows. In the interpretation of [3], rotation numbers provide information about the way orbits of a billiard flow wind around an obstacle inside a billiard table. This approach is especially useful for studying periodic billiard orbits. However, it is designed and developed for rather special billiard tables.

In the present work we introduce a certain analog of the rotation number for simply connected billiard tables. If the billiard table is multiply connected, then we develop the concept of rotation vectors. We use the billiard map, as opposed to the billiard flow; we rely on the birkhoff ergodic theorem to define the rotation vectors. Thus, in general, our approach leads to vector valued functions defined for almost all phase points. Our rotation vectors make sense for all of the basic paradigms in billiard dynamics: *elliptic, hyperbolic and parabolic* [8, 9]. For simplicity of exposition, we restrict our attention to the planar billiard dynamics. It is straightforward to extend our treatment to billiard domains on surfaces of any constant curvature [6].

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## 2. THE BILLIARD BALL PROBLEM FOR SIMPLY CONNECTED BILLIARD TABLES

Let  $\Omega \subset \mathbb{R}^2$  be a bounded, connected domain with a piecewise smooth boundary  $\partial\Omega$ . Let  $\Phi = \Phi(\Omega)$  be the *billiard map phase space*. The *billiard map*  $\varphi : \Phi \rightarrow \Phi$  has a natural invariant measure, the *liouville measure*. Let  $(s, \theta)$  be the standard coordinates in  $\Phi$  [8]. They satisfy  $0 \leq s \leq |\partial\Omega|$ , where  $|\partial\Omega|$  is the perimeter of the billiard table, and  $0 \leq \theta \leq \pi$ . The liouville measure has a density:  $d\nu = \sin \theta ds d\theta$ . Thus  $\nu(\Phi) = 2|\partial\Omega|$ . We refer to [8] and [9] for details. There is a natural projection  $p : \Phi \rightarrow \partial\Omega$ . Let  $z \in \Phi$  be a phase point. Then  $p(z) \in \partial\Omega$  is the *footpoint* of  $z$ .

From now and until the end of this section we assume that  $\Omega$  is simply connected. Thus,  $\partial\Omega$  is a simple closed curve. Choosing a reference point in  $\partial\Omega$ , endowing  $\partial\Omega$  with the positive orientation, and parameterizing  $\partial\Omega$  by the arclength, we identify  $\partial\Omega$  with the quotient  $\mathbb{R}/|\partial\Omega|\mathbb{Z}$ . We will regard  $\mathbb{R}/|\partial\Omega|\mathbb{Z}$  as a circle of perimeter  $|\partial\Omega|$ . Let  $x_1, x_2 \in \partial\Omega$  be arbitrary. Let  $0 \leq s_1, s_2 < |\partial\Omega|$  be their arclength

coordinates. Then there is a unique number  $0 \leq \xi = \xi(x_1, x_2) < |\partial\Omega|$  such that either  $s_1 + \xi = s_2$  or  $s_1 + \xi = s_2 + |\partial\Omega|$ .

Let  $z \in \Phi$ . We define the *footpoint increment function*  $\xi : \Phi \rightarrow \mathbb{R}$  via

$$(1) \quad \xi(z) = \xi(p(z), p(\varphi(z))).$$

If the limit

$$(2) \quad v(z) = \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \sum_{k=0}^{n-1} \xi(\varphi^k(z)) \right]$$

exists, we say that  $v(z)$  is the *footpoint gain* of the phase point  $z \in \Phi$ . The birkhoff ergodic theorem immediately implies the following.

**Proposition 1.** *The footpoint gain  $v(z)$  exists for almost all  $z \in \Phi$  with respect to the liouville measure. The function  $v : \Phi \rightarrow \mathbb{R}$  is measurable, invariant under  $\varphi$ , and satisfies  $0 \leq v(\cdot) \leq |\partial\Omega|$ . We have*

$$0 \leq \frac{\int_{\Phi} v(z) d\nu(z)}{\nu(\Phi)} \leq |\partial\Omega|.$$

It is often convenient to normalize the function  $v(z)$  and define the *rotation number function*  $\rho(z) = v(z)/|\partial\Omega|$ . We reformulate Proposition 1 in terms of the rotation number.

**Proposition 2.** *The rotation number  $\rho(z)$  exists for almost all  $z \in \Phi$  with respect to the liouville measure. The function  $\rho : \Phi \rightarrow \mathbb{R}$  is measurable, invariant under  $\varphi$ , and satisfies  $0 \leq \rho(\cdot) \leq 1$ . The average rotation number satisfies*

$$(3) \quad 0 \leq \frac{\int_{\Phi} \rho(z) d\nu(z)}{\nu(\Phi)} \leq 1.$$

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded, connected, simply connected domain with a piecewise smooth boundary. Let  $\varphi : \Phi \rightarrow \Phi$  be the billiard map; let  $\rho : \Phi \rightarrow \mathbb{R}$  be the rotation number function.*

1. *Let  $X \subset [0, 1]$  be a measurable set. Denote by  $1 - X \subset [0, 1]$  the set of numbers  $y = 1 - x$ , where  $x \in X$ . Then  $\rho^{-1}(X)$  and  $\rho^{-1}(1 - X)$  are measurable subsets of  $\Phi$ , and  $\nu(\rho^{-1}(X)) = \nu(\rho^{-1}(1 - X))$ .*
2. *We have*

$$(4) \quad \frac{\int_{\Phi} \rho(z) d\nu(z)}{\nu(\Phi)} = \frac{1}{2}.$$

3. *If the billiard in  $\Omega$  is ergodic then  $\rho(z) = 1/2$  for almost all  $z$ .*

*Proof.* The space  $\Phi$  carries two canonical involutions:  $\sigma : \Phi \rightarrow \Phi$  and  $\tau : \Phi \rightarrow \Phi$  [8]. Let  $z = (s, \theta) \in \Phi$ . Then  $\sigma(z) = (s, \pi - \theta)$ . In order to define  $\tau$ , we identify  $\Phi$  with the set of oriented line segments in  $\Omega$  whose endpoints belong to  $\partial\Omega$ . Then  $\tau$  corresponds to the direction reversal for the segments. Both involutions preserve the liouville measure; we have

$$\varphi = \sigma \circ \tau.$$

Therefore

$$(5) \quad \sigma \circ \varphi \circ \sigma = \tau \circ \varphi \circ \tau = \varphi^{-1},$$

i. e., both  $\sigma$  and  $\tau$  conjugate the billiard map with its inverse. By definition of the footpoint increment, we have

$$\xi(\tau(z)) = |\partial\Omega| - \xi(z).$$

This observation plus equation (5) imply the relationship

$$(6) \quad v(\tau(z)) = |\partial\Omega| - v(z).$$

Equation (6) means, in particular, that the limits in equation (2) defining  $v(z)$ ,  $v(\tau(z))$  exist or do not exist simultaneously. We rewrite equation (6) as a relationship for rotation numbers:

$$(7) \quad \rho(\tau(z)) = 1 - \rho(z).$$

Equation (7) immediately implies the first claim. The second claim is a straightforward consequence of the first. The third claim follows from the second and the birkhoff ergodic theorem.  $\blacksquare$

### 3. THE BILLIARD BALL PROBLEM FOR MULTIPLY CONNECTED BILLIARD TABLES

In this section we study the billiard dynamics in a *multiply connected planar domain*. We will use the setting and the notation established in the opening paragraph of section 2. We will say that the domain  $\Omega$  is  $q$ -connected if its boundary is a disjoint union of  $q$  connected components:

$$\partial\Omega = \partial_1\Omega \cup \cdots \cup \partial_q\Omega.$$

Thus

$$|\partial\Omega| = |\partial_1\Omega| + \cdots + |\partial_q\Omega|.$$

We choose a reference point on each component,  $\partial_\alpha\Omega$ ,  $1 \leq \alpha \leq q$ . We parameterize each component by the arclength with respect to the positive orientation. Thus, we identify each curve  $\partial_\alpha\Omega$  with the circle  $\mathbb{R}/|\partial_\alpha\Omega|\mathbb{Z}$ .

Let  $z \in \Phi$ . The footpoints  $p(z), p(\varphi(z))$  may belong to different components of  $\partial\Omega$ . In view of this observation, there is no counterpart of the footpoint increment function  $\xi(z)$  of equation (1). We will directly define the counterpart of the footpoint gain  $v(z)$  in equation (2).

We fix a boundary component  $\partial_\alpha\Omega$ . Let  $z \in \Phi$ , and let  $N \in \mathbb{N}$ . Let  $0 \leq k_0 < \dots < k_n \leq N$  be the consecutive times such that  $p(\varphi^{k_i}(z)) \in \partial_\alpha\Omega$ . Let  $0 \leq s_0, \dots, s_n < |\partial_\alpha\Omega|$  be their coordinates. For every  $0 \leq i \leq n-1$  there is a unique number  $0 \leq \xi_i < |\partial_\alpha\Omega|$  such that either  $s_i + \xi_i = s_{i+1}$  or  $s_i + \xi_i = s_{i+1} + |\partial_\alpha\Omega|$ . Suppose that the limit

$$(8) \quad v_\alpha(z) = \lim_{N \rightarrow \infty} \frac{\xi_0 + \dots + \xi_{n-1}}{N}$$

exists. Then  $v_\alpha(z)$  is the  $\alpha$ -component of the *footpoint gain vector*

$$(9) \quad \vec{v}(z) = (v_1(z), \dots, v_q(z)).$$

Let  $\vec{a} = (a_1, \dots, a_q), \vec{b} = (b_1, \dots, b_q) \in \mathbb{R}^q$  be arbitrary vectors. We will use the notation  $\vec{a} \prec \vec{b}$  to mean that  $a_\alpha \leq b_\alpha$  for  $1 \leq \alpha \leq q$ .

The birkhoff ergodic theorem immediately implies the following.

**Proposition 3.** *The footpoint gain vector  $\vec{v}(z)$  exists for almost all  $z \in \Phi$  with respect to the liouville measure. The function  $\vec{v} : \Phi \rightarrow \mathbb{R}^q$  is measurable and invariant under  $\varphi : \Phi \rightarrow \Phi$ . It satisfies*

$$\vec{0} \prec \vec{v}(\cdot) \prec (|\partial_1\Omega|, \dots, |\partial_q\Omega|).$$

We have

$$(10) \quad \vec{0} \prec \frac{\int_\Phi \vec{v}(z) d\nu(z)}{\nu(\Phi)} \prec (|\partial_1\Omega|, \dots, |\partial_q\Omega|).$$

Normalizing  $\vec{v}(\cdot)$ , we define the *rotation vector*

$$(11) \quad \vec{\rho}(z) = \left( \frac{v_1(z)}{|\partial_1\Omega|}, \dots, \frac{v_q(z)}{|\partial_q\Omega|} \right).$$

We reformulate Proposition 3 in terms of the rotation vector.

**Proposition 4.** *The rotation vector  $\vec{\rho}(z)$  exists for almost all  $z \in \Phi$  with respect to the liouville measure. The vector function  $\vec{\rho} : \Phi \rightarrow \mathbb{R}^q$  is measurable and  $\varphi$ -invariant. We have the bounds*

$$\vec{0} \prec \vec{\rho}(\cdot) \prec (1, \dots, 1).$$

*The average rotation vector satisfies*

$$(12) \quad \vec{0} \prec \frac{\int_{\Phi} \vec{\rho}(z) d\nu(z)}{\nu(\Phi)} \prec (1, \dots, 1).$$

**Theorem 2.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded, connected, multiply connected domain with a piecewise smooth boundary. Let  $\varphi : \Phi \rightarrow \Phi$  be the billiard map. Suppose that  $\Omega$  is  $q$ -connected, and let  $\vec{\rho} : \Phi \rightarrow \mathbb{R}^q$  be the rotation vector function.*

*The mean rotation vector satisfies*

$$(13) \quad \frac{\int_{\Phi} \vec{\rho}(z) d\nu(z)}{\nu(\Phi)} = \frac{1}{2} \left( \frac{|\partial_1 \Omega|}{|\partial \Omega|}, \dots, \frac{|\partial_q \Omega|}{|\partial \Omega|} \right).$$

*Suppose that the billiard in  $\Phi$  is ergodic. Then for almost all  $z \in \Phi$  we have*

$$\vec{\rho}(z) = \frac{1}{2} \left( \frac{|\partial_1 \Omega|}{|\partial \Omega|}, \dots, \frac{|\partial_q \Omega|}{|\partial \Omega|} \right).$$

*Proof.* Let  $\sigma : \Phi \rightarrow \Phi$  and  $\tau : \Phi \rightarrow \Phi$  be the canonical involutions. (See the proof of Theorem 1.) Both conjugate  $\varphi$  with  $\varphi^{-1}$ . It is useful to think of billiard orbits geometrically as oriented curves in  $\Phi$ . These curves are piecewise linear; an orbit  $\gamma$  is a sequence of straight segments with endpoints in  $\partial\Phi$ . In order to distinguish between orbits of the billiard map  $\varphi : \Phi \rightarrow \Phi$  and the geometric orbits, in what follows we will call the latter the *billiard curves*. The involutions  $\sigma$  and  $\tau$  send billiard curves into themselves, reversing the orientation. The billiard curves  $\sigma(\gamma)$  and  $\tau(\gamma)$  differ only by a shift in segment labelling. Since a finite shift in labelling of billiard curves does not change the outcome of computing footpoint gains, we do not distinguish between  $\sigma(\gamma)$  and  $\tau(\gamma)$ . We denote this infinite billiard curve by  $\tilde{\gamma}$ .

Let  $z \in \Phi$  and  $\tilde{z} \in \Phi$  be a pair of phase points related by a canonical involution. Let  $\gamma$  and  $\tilde{\gamma}$  be the corresponding billiard curves. To compute the numbers  $v_{\alpha}(z)$ ,  $v_{\alpha}(\tilde{z})$ , we perform the following operations: i) We consider finite subcurves, say  $\gamma_N$ , of  $\gamma$  (resp.  $\tilde{\gamma}_N$  of  $\tilde{\gamma}$ ) consisting of  $N$  segments; ii) We add up the footpoint gains along  $\partial_{\alpha}\Omega$  corresponding to consecutive visits of  $\gamma_N$  (resp.  $\tilde{\gamma}_N$ ) to the component  $\partial_{\alpha}\Omega$ ; iii) We divide the result by  $N$ , and take the limit  $N \rightarrow \infty$ .

Let  $\delta$  and  $\tilde{\delta}$  be the footpoint gains for  $\gamma$  and  $\tilde{\gamma}$  corresponding to the same pair of consecutive visits. Then  $\delta + \tilde{\delta} = |\partial_\alpha \Omega|$ . Therefore, the sum  $v_\alpha(z) + v_\alpha(\tilde{z})$  is equal to the product of  $|\partial_\alpha \Omega|$  and the frequency of visits to  $\partial_\alpha \Omega$  for the infinite billiard curve  $\gamma$ . Integrating this equality over  $\Phi$ , using the basic properties of  $\nu$  and the invariance of the liouville measure with respect to  $z \mapsto \tilde{z}$ , we obtain the identity

$$(14) \quad \int_{\Phi} v_\alpha(z) d\nu(z) = |\partial_\alpha \Omega|^2.$$

Putting together equations (14) for  $1 \leq \alpha \leq q$ , we arrive at the vector identity

$$(15) \quad \frac{\int_{\Phi} \vec{v}(z) d\nu(z)}{\nu(\Phi)} = \frac{1}{2} \left( \frac{|\partial_1 \Omega|^2}{|\partial \Omega|^2}, \dots, \frac{|\partial_q \Omega|^2}{|\partial \Omega|^2} \right).$$

Passing from the footpoint gain  $\vec{v}(z)$  to the rotation vector, we obtain the identity equation (13). The birkhoff ergodic theorem implies the remaining claim.  $\blacksquare$

Now we define the concept of rotation number for arbitrary connected domains.

**Definition 1.** Let  $\Omega \subset \mathbb{R}^2$  be a bounded, connected, multiply connected, in general, domain with a piecewise smooth boundary. Suppose that  $\Omega$  is  $q$ -connected, and let  $\vec{\rho} = (\rho_1, \dots, \rho_q) : \Phi \rightarrow \mathbb{R}^q$  be the rotation vector function.

We set  $\rho(z) = \rho_1(z) + \dots + \rho_q(z)$ . We call  $\rho : \Phi \rightarrow \mathbb{R}$  the rotation number function for the domain  $\Omega$ .

Note that if  $\Omega$  is simply connected, then  $\rho : \Phi \rightarrow \mathbb{R}$  in Definition 1 coincides with the rotation number that we have defined earlier. The following is an immediate consequence of Theorem 2 and Theorem 1.

**Corollary 1.** Let  $\Omega \subset \mathbb{R}^2$  be a bounded, connected domain with a piecewise smooth boundary  $\partial \Omega$  consisting of an arbitrary number of connected components. Let  $\rho : \Phi \rightarrow \mathbb{R}_+$  be the rotation number function for  $\Omega$ .

Then  $\rho$  is a measurable function with values in  $[0, 1]$ . Let  $X \subset [0, 1]$  be an arbitrary measurable set. The rotation number function satisfies the symmetry

$$\nu(\rho^{-1}(1 - X)) = \nu(\rho^{-1}(X)).$$

The mean value of  $\rho$  on  $\Phi$  with respect to the liouville measure is  $1/2$ . If the billiard in  $\Omega$  is ergodic, then  $\rho(z) = 1/2$  for  $\nu$ -almost all  $z \in \Phi$ .

#### 4. EXAMPLES, AMPLIFICATIONS, AND OPEN QUESTIONS

We will now discuss the preceding material in several special but representative cases.

**4.1. Nonergodic billiards in simply connected domains, and related questions.** Let  $\Omega \subset \mathbb{R}^2$  be a strictly convex domain with a  $C^2$  boundary. Then  $\varphi : \Phi \rightarrow \Phi$  is a  $C^1$  twist map. The rotation number  $\rho(z)$  introduced in section 2 coincides with the classical rotation number for area preserving twist maps [1]. We point out that equation (4) does not have a counterpart for general twist maps. It reflects special features of the billiard dynamics.

It is an open question whether the billiard in a domain  $\Omega$  of this kind can be ergodic [9]. If  $\partial\Omega$  is of class  $C^7$  and its curvature is strictly positive, then the billiard in  $\Omega$  is not ergodic. This follows from the results of Lazutkin [15] and R. Douady [4] about billiard caustics. If the curvature of  $\partial\Omega$  is not strictly positive, then, by a theorem of Mather [17],  $\Omega$  has no caustics. However, this does not imply the ergodicity of the billiard in  $\Omega$ . In particular, it is conceivable that the essential range of the rotation function is  $[0, 1]$ .

A caustic in  $\Omega$  yields a topologically nontrivial  $\varphi$ -invariant curve  $\Gamma \subset \Phi$ . The restriction  $\varphi|_\Gamma : \Gamma \rightarrow \Gamma$  is a circle homeomorphism. Let  $\rho(\Gamma)$  be the Poincare rotation number for  $\varphi|_\Gamma$ . Then  $\rho(z) = \rho(\Gamma)$  for all  $z \in \Gamma$ . This raises the question: for which  $\Omega$  there is an open region  $X \subset \Phi$  foliated by invariant curves? By a conjecture attributed to G.D. Birkhoff [9], the only regions satisfying this property are ellipses. The question is open except for an important special case when  $X = \Phi$ . Then, by a theorem of Bialy [2],  $\Omega$  is a disc. See also [24] for the relevant material.

Let  $\Gamma \subset \Phi$  be an invariant curve. What can we say about the rotation number  $\rho(\Gamma)$ ? Kolodziej [14] explicitly computed  $\rho(\Gamma)$  in the case when  $\Omega$  is an ellipse. Although the answer is in terms of elliptic functions, Kolodziej's approach is geometric. Namely, the work [14] crucially uses the invariant measure for a natural geometric transformation associated with a pair of nested circles.

**4.2. Ergodic billiards in simply connected domains.** By Theorem 1, ergodicity of the billiard in  $\Omega$  insures that  $\rho = 1/2$  almost everywhere. This raises the question: For which simply connected  $\Omega$  the billiard is ergodic? There is a vast literature on the subject; see [16] and the references there. As a rule, ergodic domains are not convex and their boundaries have singular points. Although there are examples of convex ergodic  $\Omega$  (e. g., the *stadium*) but  $\partial\Omega$  in these examples has



low regularity. In particular, the stadium is only  $C^1$  and not strictly convex.

Ergodicity of the billiard domains discussed in [16] is associated with the *hyperbolicity*. There are broad geometric conditions on  $\partial\Omega$  that ensure hyperbolicity [23]. However, a hyperbolic billiard domain may be non-ergodic; there are explicit examples of hyperbolic  $\Omega$  with several ergodic components [23]. It would be interesting to compute the rotation numbers for ergodic components in these examples.

There is a class of (possibly) ergodic simply connected domains with parabolic billiard dynamics. Let  $P \subset \mathbb{R}^2$  be a simply connected polygon. It is *rational* if all of its angles are commensurable with  $\pi$ . Otherwise  $P$  is *irrational*. If  $P$  is rational, then the direction of a tangent vector in  $P$  is preserved modulo the action of a finite group [7]. In view of this, the billiard phase space is foliated by a one-parameter family of invariant subsets, hence the billiard in a rational polygon  $P$  is not ergodic. If  $P$  is irrational, then there is no obvious obstruction to ergodicity. Presently there are no examples of nonergodic irrational polygons [9]. A theorem of Vorobets [22] says that if the angles of an irrational polygon  $P$  are extremely fast approximated by  $\pi$ -rational numbers, then  $P$  is ergodic. This result yields explicit examples of simply connected (even convex) ergodic polygons. By Theorem 1, for such  $P$  we have  $\rho = 1/2$  almost everywhere. It would be interesting to compute the function  $\rho(z)$  for rational polygons. The case of *arithmetic* polygons [10] seems especially intriguing.

**4.3. Billiards in multiply connected domains.** Multiply connected billiard domains arise naturally in several contexts. We will discuss one of them. Let  $\gamma \subset \mathbb{R}^2$  be a closed, convex curve. The *string construction* [11] yields a one-parameter family  $X = X(\gamma, l)$  of convex domains containing  $\gamma$  in their interior. The parameter  $l > |\gamma|$  is the string length. The family  $X(\gamma, l)$  consists of planar domains having  $\gamma$  as a billiard caustic [11]. Let  $\Phi(l)$  be the phase space for the billiard map in  $X(\gamma, l)$ ; let  $\Gamma(l) \subset \Phi(l)$  be the invariant curve corresponding to  $\gamma$ ; let  $\rho(l)$  be its rotation number. The monotonic function  $l \mapsto \rho(l)$  yields the so called *phase locking* phenomenon [12]. The case when  $\gamma$  is a polygon is especially interesting. Then the domains  $X = X(\gamma, l)$  exhibit the phenomenon of *instability of the boundary* [13], [11]. This means that near  $\partial X$  there is an open region in  $X$  which is *free of caustics*. It is plausible that  $\gamma \subset X(\gamma, l)$  is the *last caustic* [12].

Let  $\Omega = X(\gamma, l) \setminus \text{interior}(\gamma)$ . Thus  $\Omega = \Omega(\gamma, l)$  is a multiply connected domain. It would be instructive to investigate the rotation

vector for the billiard in  $\Omega$ . See section 3. The case when  $\gamma$  is a circle is especially simple. Then the regions  $\Omega(\gamma, l)$  are the rotationally symmetric annuli. The billiard in a rotationally symmetric annulus is completely integrable; concentric circles yield invariant curves that foliate the phase space. When  $\gamma$  is an ellipse, the situation is more complicated, but the billiard in question is still completely integrable. There are two kinds of invariant curves: confocal ellipses and confocal hyperbolas. It is plausible that these are the only multiply connected domains with completely integrable billiard dynamics. This is a natural extension of the conjecture attributed to G.D. Birkhoff. See section 4.1.

The above multiply connected domains can be described as follows:  $\Omega$  is the annulus between a closed convex curve  $\gamma$  and the boundary of a billiard table, say  $X$ , having  $\gamma$  as a billiard caustic. An interesting class of multiply connected domains arises when we perturb this situation in a particular way. Namely,  $\Omega$  is obtained by moving  $\gamma$  inside  $X$  by a small planar isometry. In the simplest case, when  $\gamma$  is a circle, we obtain annular regions between two non-concentric circles. We will refer to these regions as *asymmetric annuli*. Let  $\Omega$  be an asymmetric annulus; let  $\Phi$  be the corresponding billiard phase space. There is an obvious invariant region  $\Phi_0 \subset \Phi$  foliated by invariant curves. The restriction of billiard dynamics to  $\Phi_0$  is completely integrable. The analysis of hyperbolicity and ergodicity in the complement  $\Phi \setminus \Phi_0$  by currently available methods is inconclusive. However, numerical simulations indicate that the billiard dynamics in  $\Phi \setminus \Phi_0$  is chaotic.

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